


Gauss Law

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Electric Flux

- **Electric flux** φ is the drift of the electric field through a certain surface. Mathematically it is defined as closed integral of the scalar product of electric field and surface area

$$\varphi = \oint \vec{E}.d\vec{A}$$

- The enclosed surface is called **Gaussian Surface**. For uniform area, the above integral tends to be

$$\varphi = \vec{E}.\vec{A}$$

- **The electric flux** through a Gaussian surface is proportional to the electric field lines passing through that surface.
- **Zero flux** occurs when the surface is normal to the electric field lines.

Gauss Law

- **Gauss law** states that the electric flux over a closed surface is proportional to the total electric charge contained in the surface, q_{enc}.

$$\varphi = \frac{q_{enc}}{\varepsilon_0}$$

- The above equation suggests that the electric flux **does not depend** on the shape of the surface.
- The **Gauss law** can be written as

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

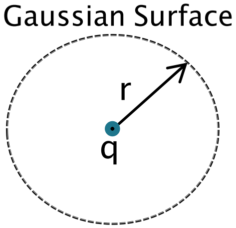
- The unit of electric flux is $N.m^2/C$.

Applications of Gauss Law- Point Charge

Suppose we have a point charge q, the electric field at any point r from the charge can be evaluated from Gauss law

$$\oint \vec{E}.\,d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

First take a Gaussian surface ‘sphere’ of radius r



Using the above formula with $q_{enc} = q$, we get

$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

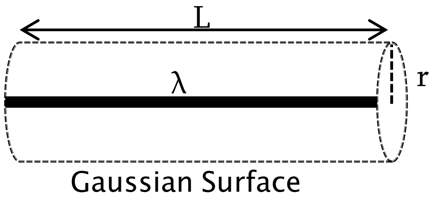
$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{k\,q}{r^2}$$

Applications of Gauss Law- Line Charge

Suppose we have a line of charge with λ , the electric field at any point r from the line can be evaluated from Gauss law

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

First take a Gaussian surface 'cylinder' of radius r and length L



Using the above formula with $q_{enc} = \lambda L$, we get

$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} \quad \rightarrow \quad E(2\pi r L) = \frac{\lambda L}{\varepsilon_0}$$

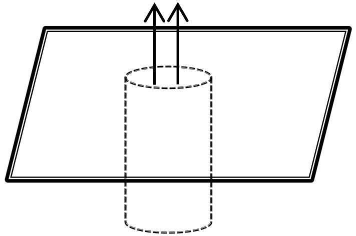
$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Applications of Gauss Law- Infinite Sheet

Suppose we have an infinite **conducting** sheet having surface charge σ, the electric field at any point r from the sheet can be evaluated from Gauss law

$$\oint \vec{E}.\, d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

Conducting sheet

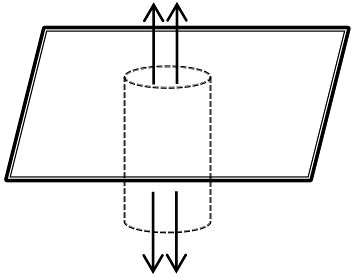


First take a Gaussian surface 'cylinder' of cross-section area A

Using the above formula with $q_{enc} = \sigma A$, we get

$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \quad \rightarrow \quad E = \frac{\sigma}{\varepsilon_0}$$

Non-conducting sheet



If the sheet is **non-conducting**, the electric field is

$$EA + EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \quad \rightarrow \quad E = \frac{\sigma}{2\varepsilon_0}$$

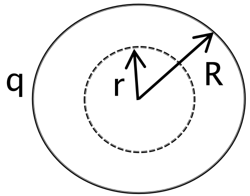
Applications of Gauss Law- Conducting Sphere

Suppose we have a conducting sphere of radius R and charge q, the electric field at any point r from the center can be evaluated from Gauss law

$$\oint \vec{E}.d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

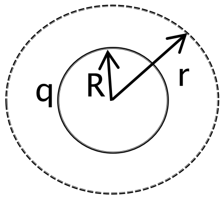
We have two region to find the electric field at, inside and outside the sphere. Let us determine the field inside the sphere. Take a Gaussian surface 'sphere' of radius r. For any conducting sphere, **the charge is distributed over its surface** and no charge is enclosed within it. Therefore using the above formula with $q_{enc} = 0$, we get

$$E = 0$$



Applications of Gauss Law- Conducting Sphere

Now let us determine the field outside the sphere. Take a Gaussian surface 'sphere' of radius r



The charge enclosed within the Gaussian surface is q. Therefore using the above formula with $q_{enc} = q$, we get

$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{k\,q}{r^2}$$

Therefore, **the electric field just about the surface of the sphere**, $r = R$, is

$$E = \frac{k\,q}{R^2}$$

Applications of Gauss Law- Insulating Sphere

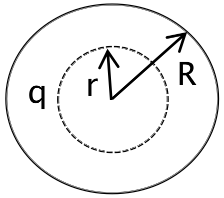
Suppose we have an insulating sphere of radius R and charge q, the electric field at any point r from the center can be evaluated from Gauss law

$$\oint \vec{E}.\,d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

We have two region to find the electric field at, inside and outside the sphere. Let us determine the field inside the sphere. Take a Gaussian surface ‘sphere’ of radius r. For any insulating sphere, the charge is distributed throughout the volume and no charge is located over surface. Please note that the enclosed charge is not q but a part of it. Therefore we have to find this charge firstly

$$q_{enc} = \left(\frac{Volume\ of\ Gaussian\ sphere}{Volume\ of\ original\ sphere}\right) q = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right) q = \left(\frac{r^3}{R^3}\right) q$$

Now using the Gauss law with $q_{enc} = \left(\frac{r^3}{R^3}\right) q$, we get

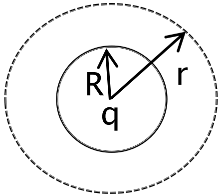


$$E(4\pi r^2) = \frac{\left(\frac{r^3}{R^3}\right) q}{\varepsilon_0}$$

$$E = \frac{q\,r}{4\pi\varepsilon_0 R^3}$$

Applications of Gauss Law- Insulating Sphere

Now let us determine the field outside the sphere. Take a Gaussian surface 'sphere' of radius r. The charge enclosed within the Gaussian surface is now q. Therefore using the Gauss law with $\mathrm{q_{enc}} = q$, we get



$$EA = \frac{\mathrm{q_{enc}}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{k\,q}{r^2}$$

Therefore, **the electric field just about the surface of the sphere**, $r = R$, is

$$E = \frac{k\,q}{R^2}$$

Summary of Laws-1

- Electric field at any point r due to **a point charge** q is

$$E = \frac{k\,q}{r^2}$$

- Electric field at any point r **due to a line of charge** of charge λ is

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

- Electric field at any point r **due to a conducting sheet** of charge σ is

$$E = \frac{\sigma}{\varepsilon_0}$$

- Electric field at any point r **due to a non-conducting sheet** of charge σ is

$$E = \frac{\sigma}{2\varepsilon_0}$$

Summary of Laws-2

- Electric field at any point r **inside a conducting sphere** of radius R and charge q is

$$E = 0$$

- Electric field at any point r **outside a conducting sphere** of radius R and charge q is

$$E = \frac{k\, q}{r^2}$$

- Electric field **on the surface of a conducting sphere** of radius R and charge q is

$$E = \frac{k\, q}{R^2}$$

- Electric field at any point r **inside an insulating (solid) sphere** of radius R and charge q is

$$E = \frac{q\, r}{4\pi\varepsilon_0 R^3}$$

- Electric field at any point r **outside an insulating sphere** of radius R and charge q is

$$E = \frac{k\, q}{r^2}$$

- Electric field **on the surface of an insulating sphere** of radius R and charge q is

$$E = \frac{k\, q}{R^2}$$

Worked Exercises

1. Two charges 25.9 µC and -8.2 µC are confined in a spherical surface of radius 5 cm. Calculate the net electric flux though the surface. From this calculate the magnitude of the electric field at that point.

Solution

The electric flux is defined as

$$\varphi = \frac{q_{enc}}{\varepsilon_0} = \frac{(25.9 - 8.2) \times 10^{-6}}{8.85 \times 10^{-12}} = 2.0 \times 10^{6} \mathrm{N.m^2/C}$$

The electric field can be calculated from

$$\varphi = \vec{E}.\vec{A}$$

But the electric field at each point is parallel to the area section, therefore

$$\varphi = EA \qquad \rightarrow \qquad E = \frac{\varphi}{A} = \frac{2.0 \times 10^{6}}{4\pi \times 0.05^2} = 6.4 \times 10^{7} \mathrm{N/C}$$

Worked Exercises

2. A certain charge Q is enclosed in a sphere of radius R. If the electric flux through the sphere's surface is 450 $\mathrm{N.m^2/C}$, calculate the charge Q.

Solution

The electric flux is defined as

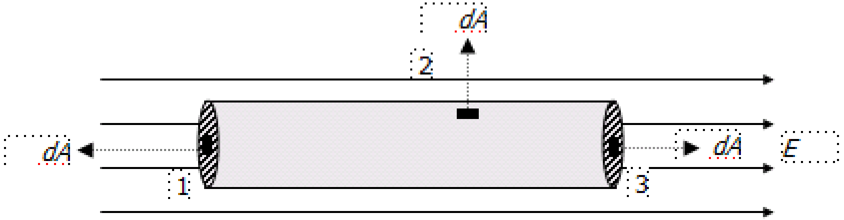
$$\varphi = \frac{q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

$$Q = \varepsilon_0 \varphi = 8.85 \times 10^{-12} \times 450 = 3\ \mathrm{nC}$$

Worked Exercises

3. A cylinder of radius 2 cm is horizontally placed in a uniform electric field of 2000 N/C. Calculate the net electric flux through the cylinder.

Solution



As shown above, we divided the cylinder into 3 faces. Through face 1, we note that the area is anti-parallel to the electric field (angle is 180). Therefore

$$\varphi_1 = \vec{E}.\vec{A} = EA\,cos180 = -EA$$

Worked Exercises

Through the face 2, we note that the area is normal to the electric field (angle is 90). Therefore

$$\varphi_2 = \vec{E}.\vec{A} = EA\ cos90 = 0$$

Through the face 3, we note that the area is parallel to the electric field (angle is 0). Therefore

$$\varphi_3 = \vec{E}.\vec{A} = EA\ cos0 = EA$$

The net electric flux through the cylinder is

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 \ = -EA + 0 - EA = 0$$

Worked Exercises

4. An 8-m^2 plate is immersed in a uniform electric field of 2000 N/C. If the plane of the plate makes an angle of 75^0 with the electric field, calculate the electric flux and then find the enclosed charge.

Solution

The electric flux is defined as

$$\varphi = \vec{E}.\vec{A} = EA\ \cos\theta = 8 \times 2000 \times \cos 75 = 4.14 \times 10^3\ \mathrm{N.m^2/C}$$

The enclosed charge can be evaluated from

$$\varphi = \frac{q_{enc}}{\varepsilon_0} \quad \rightarrow \quad q_{enc} = \varepsilon_0 \varphi = 8.85 \times 10^{-12} \times 4.14 \times 10^3 = 0.36\ \mathrm{nC}$$

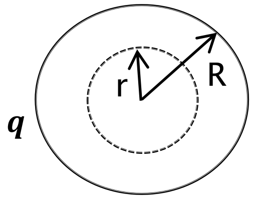
Worked Exercises

5. A metallic sphere of radius 5cm carrying a charge of $q = 5\ \mu C$. Calculate the magnitude of the electric field at (i) 3 cm and (ii) 10 cm from the center.

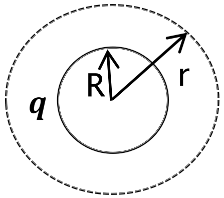
Solution

(i) At 3 cm from the center- inside the sphere- we have $q_{enc} = 0$, therefore the electric field is

$$E = 0$$



(ii) At 10 cm from the center- outside the sphere- we have $q_{enc} = q$, therefore the electric field is



$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

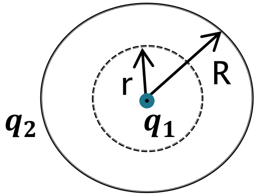
$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{k\,q}{r^2} = \frac{9 \times 10^9 \times 5 \times 10^{-6}}{0.1^2} = 4.5 \times 10^6 \text{N/C}$$

Worked Exercises

6. A charges of q_1 =2 μC is surrounded by a conducting sphere of radius 5cm carrying a charge of $q_2 = 5$ μC. Calculate the magnitude of the electric field at (i) 3 cm and (ii) 10 cm from the center.

Solution

(i) At 3 cm from the center- inside the sphere- we have $q_{enc} = q_1$, therefore the electric field is

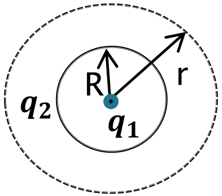


$$EA = \frac{q_{enc}}{\varepsilon_0} = \frac{q_1}{\varepsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{q_1}{\varepsilon_0}$$

Worked Exercises

$$E = \frac{q_1}{4\pi\varepsilon_0 r^2} = \frac{k\, q_1}{r^2} = \frac{9 \times 10^9 \times 2 \times 10^{-6}}{0.03^2} = 2 \times 10^7 \text{N/C}$$

(ii) At 10 cm from the center- outside the sphere- we have $\text{q}_{\text{enc}} = q_1 + q_2$, therefore the electric field is



$$EA = \frac{\text{q}_{\text{enc}}}{\varepsilon_0} = \frac{q_1 + q_2}{\varepsilon_0} \quad \rightarrow \quad E(4\pi r^2) = \frac{q_1 + q_2}{\varepsilon_0}$$

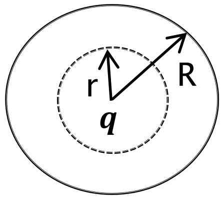
$$E = \frac{q_1 + q_2}{4\pi\varepsilon_0 r^2} = \frac{k\,(q_1 + q_2)}{r^2} = \frac{9 \times 10^9 \times (2 + 5) \times 10^{-6}}{0.1^2} = 6.3 \times 10^6 \text{N/C}$$

WORKED EXERCISES

7. A solid sphere of radius 5cm carrying a charge of $q = 5$ μC. Calculate the magnitude of the electric field at (i) 3 cm and (ii) 10 cm from the center.

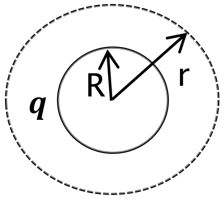
Solution

(i) At 3 cm from the center- inside the sphere- we have $q_{enc} = \left(\frac{r^3}{R^3}\right) q$, therefore the electric field is



$$E = \frac{q\,r}{4\pi\varepsilon_0 R^3} = \frac{5 \times 10^{-6} \times 0.03}{4\pi \times 8.85 \times 10^{-12} \times 0.05^3} = 1.08 \times 10^{7}\,\mathrm{N/C}$$

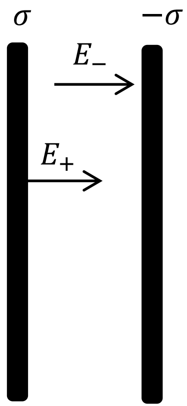
(ii) At 10 cm from the center- outside the sphere- we have $q_{enc} = q$, therefore the electric field is



$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{5 \times 10^{-6}}{4\pi \times 8.85 \times 10^{-12} \times 0.1^2} = 4.5 \times 10^{6}\,\mathrm{N/C}$$

WORKED EXERCISES

8. Two parallel conducting sheets carry equal but opposite surface charges of $8.85\ \mathrm{nC/m^2}$. Calculate the electric field between them.



Solution

The electric field due to a conducting sheet is

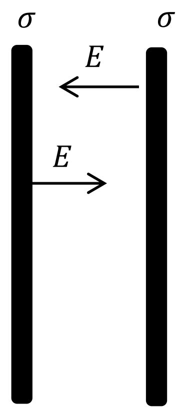
$$E = \frac{\sigma}{\varepsilon_0}$$

From the diagram we note that the direction of both electric fields are same, therefore the magnitude of E is

$$E = E_- + E_+ = \frac{\sigma}{\varepsilon_0} + \frac{\sigma}{\varepsilon_0} = \frac{2\sigma}{\varepsilon_0} = \frac{2 \times 8.85 \times 10^{-9}}{8.85 \times 10^{-12}} = 2000\ \mathrm{N/C}$$

Worked Exercises

9. Two parallel conducting sheets carry equal surface charges of $8.85\ \mathrm{nC/m^2}$. Calculate the electric field between them.



Solution

The electric field due to a conducting sheet is

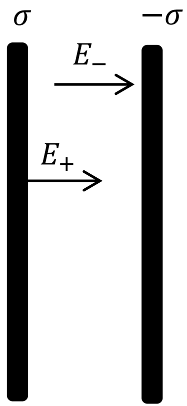
$$E = \frac{\sigma}{\varepsilon_0}$$

From the diagram we note that the direction of each electric field opposes the other, therefore

$$E = E - E = 0$$

Worked Exercises

10. Two parallel non-conducting sheets carry equal but opposite surface charges of $8.85\ \mathrm{nC/m^2}$. Calculate the electric field between them.



Solution

The electric field due to a non conducting sheet is

$$E = \frac{\sigma}{2\varepsilon_0}$$

From the diagram we note that the direction of both electric fields are same, therefore the magnitude of E is

$$E = E_- + E_+ = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0} = \frac{8.85 \times 10^{-9}}{8.85 \times 10^{-12}} = 1000\ \mathrm{N/C}$$

WORKED EXERCISES

11. An electron is placed near a non-conducting sheet carrying a surface charge density of 17.7nC/m^2. Calculate the magnitude of the electric force acting on the electron.

Solution

The electric field due to a non-conducting sheet is

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{17.7 \times 10^{-9}}{2 \times 8.85 \times 10^{-12}} = 1000 \text{ N/C}$$

Hence the magnitude of the electric force on the electron is

$$F = eE = 1.6 \times 10^{-19} \times 1000 = 1.6 \times 10^{-16} \text{N}$$